

A Harris-Kesten theorem for confetti percolation

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Abstract

Percolation properties of the dead leaves model, also known as confetti percolation, are considered. More precisely, we prove that the critical probability for confetti percolation with square-shaped leaves is $1/2$. This result is related to a question of Benjamini and Schramm concerning disk-shaped leaves and can be seen as a variant of the Harris-Kesten theorem for bond percolation. The proof is based on techniques developed by Bollobás and Riordan to determine the critical probability for Voronoi and Johnson-Mehl percolation.

1 Introduction

In recent years much progress has been made to determine the critical value of various two-dimensional configuration models in percolation theory and statistical mechanics that exhibit a more complex dependency structure than classical Bernoulli percolation (see e.g. [4, 6, 2, 16, 1]). In this paper we consider a spatial percolation process based on the so-called *dead leaves model* which is popular in stochastic geometry (see e.g. [13]). This model describes the coloring of \mathbb{R}^2 observed when covering the plane by black and white leaves according to a space-time Poisson process (see Section 2 for a precise definition). In percolation literature this process is also known under the name of *confetti percolation* and Benjamini and Schramm have conjectured in [3, Problem 5] that $p_c = 1/2$ for the case of disk-shaped leaves. We will show how the techniques from [4] and [7] can be used to prove that the critical probability for square-shaped confetti percolation is precisely $1/2$.

Let us give a rough outline of the main ideas. As in Bernoulli percolation the part $p_c \geq 1/2$ follows from Zhang's elegant proof of $\theta(1/2) = 0$. Indeed to apply his method we need to check positive correlation of black-increasing events (this is standard, see e.g. [15, 5]) and the uniqueness of the infinite cluster. For the latter part one may use the geometric method of [10] which has the advantage that no additional discretization is needed.

Proving $p_c \leq 1/2$ is less canonical and we follow the framework developed by Bollobás and Riordan in [4, 7]. Although one might hope at first that the problem of confetti percolation is considerably simpler than the problem of Voronoi percolation (for instance since the range of dependence is finite), still a fair amount of work needs to be done to resolve discretization issues.

In Section 2 we give detailed definitions of planar confetti-type percolation models and introduce further useful notation. The proof of $\theta(1/2) = 0$ is provided in Section 3. The proof of $p_c \leq 1/2$ can be subdivided into two steps that are presented in Sections 4 and 5. In Section 4 we show that the assumptions of the general RSW-type theorem of Bollobás and Riordan (see e.g. [4]) are satisfied in the confetti model. Together with the sharp-threshold theorem [8,

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Lemma 1] (which itself is a variant of the sharp-threshold result [9, Theorem 2.1]) this result is used to complete the proof of $p_c \leq 1/2$ in Section 5 (except for the proof of a rather complicated elementary geometric lemma which is postponed to Section 6)

We believe that a similar approach could be used to consider disk-shaped leaves, although starting from the current article this seems not completely straightforward. Furthermore we would be very interested in considering generalizations using different shapes of sufficiently symmetric leaves or where the leaves are rotated at random. A very interesting idea to make the current argument less dependent on the specific shape of the leaf was suggested by an anonymous referee in Remark 1. However, this approach depends on an estimate on the tail-behavior of the number of visible leaves of the dead leaves model in a bounded sampling window which seems to be not completely trivial to establish.

2 Notations and basic definitions

For $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $\{A_s\}_{s \in [0, \infty)}$ a family of events, we say that A_s holds with high probability (short whp) if $\mathbb{P}(A_s) \rightarrow 1$ as $s \rightarrow \infty$. For $\rho > 0$, $u \in \mathbb{R}^2$ we denote by $Q_\rho(u) = u + \rho[-1/2, 1/2]^2$ the square of side length ρ centered at u and write $Q(u) = Q_1(u)$.

For $\varphi = \{x_n\}_{n \geq 1} = \{(z_n, t_n, \sigma_n)\}_{n \geq 1} \subset \mathbb{R}^2 \times [-1, \infty) \times \{\pm 1\}$ locally finite we will often use the notation $y_n = (z_n, t_n)$ to denote the space-time coordinates of the element (z_n, t_n, σ_n) . Furthermore we denote by $A \subset \mathbb{R}^2$ a leaf which is compact, path-connected, contains the origin o in its interior, satisfies $A = -A$ and whose boundary is rectifiable. Then the dead leaves process describes a sequence of colored leaves falling onto the plane according to the space-time process φ . To be more precise, at time t_n a leaf of shape A with center z_n and of color σ_n (say black if $\sigma_n = 1$ and white if $\sigma_n = -1$) appears. This yields a coloring of the plane by defining the color of a point $u \in \mathbb{R}^2$ to be the color of the first leaf covering u (or undefined if there is either no such leaf or if it is non-unique). Note that we observe the configuration of leaves *from below* in order to obtain a static coloring of the entire plane.

To each such locally finite φ we can associate a function $height_\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ mapping a point $u \in \mathbb{R}^2$ to the time the leaf visible at u had arrived. Formally we put $height_\varphi(u) = \min(\{t_m : x_m = (z_m, t_m, \sigma_m) \in \varphi, u - z_m \in A\})$ if this set is non-empty and the minimum is unique, and $height_\varphi(u) = -2$ otherwise. To each point of \mathbb{R}^2 we assign a number from $\{0, \pm 1\}$ according to the function $\psi_\varphi : \mathbb{R}^2 \rightarrow \{0, \pm 1\}$ defined by $\psi(u) = \psi_\varphi(u) = \sigma_n$ where the index n is chosen so that $height_\varphi(u) = t_n$ and $\psi_\varphi(u) = 0$ if $height_\varphi(u) = -2$. Sometimes we also write ψ -black to describe the attribute of being black in the coloring ψ . For instance the connected components of $\psi^{-1}(1)$ are called *ψ -black connected components*. Furthermore, for colorings $\psi_1, \psi_2 : \mathbb{R}^2 \rightarrow \{\pm 1\}$ we say ψ_1 *black-dominates* ψ_2 if $\psi_1(x) \geq \psi_2(x)$ holds for all $x \in \mathbb{R}^2$. For $\varphi_1, \varphi_2 \subset \mathbb{R}^2 \times [-1, \infty) \times \{\pm 1\}$ locally finite we also say that φ_1 *black-dominates* φ_2 if ψ_{φ_1} black-dominates ψ_{φ_2} .

One can add a probabilistic flavor to this model by replacing the locally finite set φ by an independently $\{\pm 1\}$ -marked homogeneous Poisson point process $X \subset \mathbb{R}^2 \times [0, \infty)$. Furthermore we write $p = \mathbb{P}(\sigma_n = 1)$ for the probability that a fixed leaf is colored black in X . It is easy to see that in the coloring ψ_X with probability 1 all points of \mathbb{R}^2 are colored either black or white. We write $\theta(p, A)$ for the probability that the origin is contained in an unbounded ψ -black component. In case that the leaf $A = Q(o)$ is the unit square centered at the origin we also write $\theta(p)$ for $\theta(p, A)$. Furthermore we use the standard definition of critical probability for percolation, namely $p_{c,A} = \inf\{p > 0 : \theta(p, A) > 0\}$ and write p_c in the special case $A = Q(o)$. Note also that it is straightforward to extend the definition of the confetti process so as to allow leaves of randomly varying size and shape.

3 $\theta(1/2) = 0$

3.1 Harris's inequality

The basic statement of Harris's inequality is that black-increasing events are positively correlated. Although the classical Harris inequality is stated in a lattice setting, some extra technical work makes it possible to adapt it to the situation of confetti percolation. Similar technical adjustments are explained in detail by Bollobás and Riordan in [5] for the case of Voronoi percolation and we follow their presentation.

An event E that is defined in terms of two independent Poisson processes (X^+, X^-) is called *black-increasing* if for every configuration $\omega_1 = (X_1^+, X_1^-)$ and $\omega_2 = (X_2^+, X_2^-)$ with $X_1^+ \subset X_2^+$, $X_1^- \supset X_2^-$ and $\omega_1 \in E$ we have $\omega_2 \in E$. Similarly one defines the notion of black-increasing functions.

First let us consider Harris's inequality when only the process X^+ is involved. Let $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$ be the canonical probability space of the random variable X^+ . Denote by Σ_k the σ -algebra generated by the following information. Set $n = 2^k$ and divide $[-n, n]^2 \times [0, n]$ in $4n^6$ cubes of side length $1/n$. Decide for each of them whether it contains at least one point of X^+ or not. The local finiteness of X^+ then implies that $\{\Sigma_k\}_{k \geq 1}$ forms a filtration of $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$.

Let g_1, g_2 be increasing, bounded and measurable functions. As $\mathbb{E}(g_i | \Sigma_k)$ is an increasing function on the discrete product space determined by Σ_k , the lattice version of Harris's inequality implies $\mathbb{E}(\mathbb{E}(g_1 | \Sigma_k) \mathbb{E}(g_2 | \Sigma_k)) \geq \mathbb{E}(g_1) \mathbb{E}(g_2)$. As g_1, g_2 are bounded, the martingale convergence theorem implies $\mathbb{E}(g_i | \Sigma_k) \xrightarrow{k \rightarrow \infty} g_i$ and dominated convergence yields $\mathbb{E}(g_1 g_2) \geq \mathbb{E}(g_1) \mathbb{E}(g_2)$. Starting from this result it is easy to derive Harris's inequality for the case of confetti percolation (we omit the standard proof and refer the reader to [5] for details in the closely related Poisson-Voronoi case).

Lemma 1. *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the canonical product space of (X^+, X^-) and let $A, B \in \mathcal{A}$ be two increasing events. Then $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) \mathbb{P}(B)$.*

In particular if $X \subset \mathbb{R}^2 \times [0, \infty) \times \{\pm 1\}$ is an independently $\{\pm 1\}$ -marked homogeneous Poisson point-process, then Lemma 1 can be applied to $X^+ = \{y_n : \sigma_n = 1\}$ and $X^- = \{y_n : \sigma_n = -1\}$.

3.2 Uniqueness of the infinite cluster

Let $A \subset \mathbb{R}^2$ denote a leaf as described in Section 2 and which is additionally invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. In this subsection we note that uniqueness of the infinite cluster for confetti percolation is a corollary from the results of Gandolfi, Keane and Russo developed in [10]. Indeed their method is based purely on geometric properties of \mathbb{R}^2 (such as the Jordan curve theorem) and works just as well in continuous situations. A similar (but in fact more complicated) adaption was considered in [12]. In order to deduce uniqueness of the infinite cluster we only need to check that the percolation process under consideration satisfies the following assumptions.

- (i) the origin lies in the interior of a cluster a.s. and it lies in the interior of a white cluster with positive probability
- (ii) \mathbb{P} is invariant under horizontal and vertical translations, under rotations by $\pi/2$ and under reflections at the coordinate axes
- (iii) \mathbb{P} is ergodic with respect to (discrete) horizontal translations and, separately, with respect to (discrete) vertical translations
- (iv) black-increasing events are positively correlated
- (v) there exists at least one infinite cluster with positive probability

Note that confetti percolation always satisfies the first four of these items so that we obtain the following result.

Proposition 2. *Let $0 < p < 1$ and denote by N the (random) number of unbounded black clusters. Then $\mathbb{P}_p(N = 1) = 1$ or $\mathbb{P}_p(N = 0) = 1$.*

3.3 Zhang's theorem

Zhang's famous proof of $\theta(1/2) = 0$ is based on certain symmetry requirements of the percolation model (invariance under horizontal and vertical translations as well as under rotations by $\pi/2$), the uniqueness of the unbounded cluster, positive correlation of black-increasing events, the property that black and white components cannot cross and the symmetry between black and white coloring. For details one may consult the standard textbooks [11] and [5] (both expositions are very similar although [11] does not use the Bernoulli hypothesis which makes slightly more suitable for our specific application). All these assumptions are true in the case of confetti percolation and we obtain the following result.

Proposition 3. *Let A denote a fixed leaf that is invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. Then $\theta(1/2, A) = 0$.*

4 Checking the conditions of the RSW theorem

Let again $A \subset \mathbb{R}^2$ denote a leaf as described in Section 2 which is additionally invariant under rotations by $\pi/2$ and under reflections at the coordinate axes. The first step in the proof of $p_c \leq 1/2$ is to check the validity of the assumptions of the general RSW-type theorem developed by Bollobás and Riordan in [4] (see also [6, Section 5]). For $R \subset \mathbb{R}^2$ a rectangle we denote by $H(R)$ the event that there exists a black horizontal crossing of R . The goal of this section is to prove the following result.

Proposition 4. *Let $0 < p < 1$, $\rho > 1$ be arbitrary and suppose there exists $c > 0$ such that $\mathbb{P}(H([0, s] \times [0, s])) \geq c$ holds for all sufficiently large $s > 0$. Then there exists $c' > 0$ such that for all $s_0 > 0$ we can find $s \geq s_0$ with $f_p(\rho, s) = \mathbb{P}_p(H([0, \rho s] \times [0, s])) \geq c'$.*

As explained in [6, Section 5] to obtain Proposition 4 for a given percolation model in \mathbb{R}^2 only four basic assumptions have to be checked.

- (i) black increasing events are positively correlated (we have seen this in Section 3.1)
- (ii) the model has the symmetries of \mathbb{Z}^2 (true, since our leaves have this property)
- (iii) disjoint regions are asymptotically independent (in fact for the dead leaves model the range of dependence is finite)
- (iv) for any fixed rectangle R there exists $C > 0$ such that the probability that $H(sR)$ holds but the shortest black horizontal crossing of sR has length $> s^C$ tends to 0 as $s \rightarrow \infty$ (this is not clear *a priori* – the remainder of this section is devoted to this item)

To prove useful asymptotics for the length of a shortest black horizontal crossing we observe that the length of such a crossing is bounded from above by the perimeter of the black connected component in which it is contained. Since the boundary of this component is a subset of the union of the boundary of R and the union of the boundaries of (partly) visible leaves intersecting R , it suffices to bound the latter.

Choose $r_1, r_2 > 0$ such that $Q_{r_1}(o) \subset A \subset Q_{r_2}(o)$. The first step is to give a more economical construction of the dead leaves process on a rectangle of the form $R = [0, as] \times [0, bs]$ for $a, b, s > 0$. Define $R^{\text{ext}} = [-r_2, as + r_2] \times [-r_2, bs + r_2]$ and denote by $c(s)$ a function converging to a constant $c > 0$ to be determined in the following paragraph. Furthermore let

$X = \{(y_n, \sigma_n)\}_{1 \leq n \leq N} \subset R^{\text{ext}} \times [0, c(s) \log s] \times \{\pm 1\}$ be an independently $\{\pm 1\}$ -marked homogeneous Poisson point process with intensity 1 whose marks are iid, have constant shape A and satisfy $\mathbb{P}(\sigma_n = 1) = p$. Furthermore denote by $\pi_1 : R^{\text{ext}} \times [0, c(s) \log s] \times \{\pm 1\} \rightarrow R^{\text{ext}}$ the projection onto the first coordinate. Then $Z = \pi_1(X) \subset R^{\text{ext}}$ is a homogeneous Poisson point process on R^{ext} with intensity $\lambda = c(s) \log s$. Note that to obtain from X a realization of the dead leaves process on the finite rectangle R it suffices to simulate a further $\{\pm 1\}$ -marked homogeneous Poisson process $X' \subset R^{\text{ext}} \times [c(s) \log s, \infty) \times \{\pm 1\}$ and consider the coloring of R induced by the superposition of X and X' .

Let us denote by $B_s^{(1)}$ the event $R \subset \bigcup_{z \in Z} \{z + A\}$. Observe that if $B_s^{(1)}$ occurs then the process X' is not needed to determine the coloring of R induced by $X \cup X'$. We claim that $\mathbb{P}(B_s^{(1)}) \rightarrow 1$ as $s \rightarrow \infty$. Using the notation $M = \{v \in (r_1/4)\mathbb{Z}^2 : (v + [0, r_1/4]^2) \cap R \neq \emptyset\}$ we see that $B_s^{(1)}$ is implied by the occurrence of the event $Z(v + [0, r_1/4]^2) \geq 1$ for all $v \in M$. However the probability of its complement can be bounded from above by $|M| \exp(-\lambda r_1^2/16) \leq 16r_1^2(as + 2r_2)(bs + 2r_2) \exp(-\lambda r_1^2/16)$. Choosing a suitable $c(s)$ for which $s^2 \exp(-\lambda r_1^2/16) \in O(s^{-1})$ (e.g., for $A = Q(o)$ we choose $c(s) = 50 \lfloor \log s \rfloor / \log s$), we see that this expression tends to 0 as $s \rightarrow \infty$. In particular, we conclude that the number of visible squares intersecting R is of order $O(s^3)$ whp thereby checking the final assumption in the RSW-type result.

5 $p_c \leq 1/2$

In the following we restrict our attention to square-shaped leaves, i.e. $A = Q(o)$. In this section we show how the RSW-type theorem of Section 4 in conjunction with a suitable sharp-threshold result can be used to obtain $\theta(p) > 0$ for $p > 1/2$. As in [5, Section 8.3] we see that in order to apply this sharp threshold result, we need to work with a mesh-size $\delta(s)$ of the form $\delta(s) \sim s^{-\gamma}$ for some constant $\gamma > 0$. Since $\gamma > 0$ may be quite small we have to expect the occurrence of discretization defects. For simplicity of exposition we assume from now on that $s > 0$ is an integer. Let us denote by $\mathbb{T}^2 = \mathbb{T}_{10s}^2$ the two-dimensional torus obtained by the standard glueing of the boundaries of $[0, 10s]^2$. Furthermore, it is no problem to construct a confetti process on the torus in the same way as we did for the case of \mathbb{R}^2 . Even the more economical construction of the process can be transferred. Indeed, let us write $\lambda = \lambda(s) = 50 \lfloor \log s \rfloor$ and let $X \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ be an independently $\{\pm 1\}$ -marked homogeneous Poisson point process with intensity 1 whose marks have constant shape $A = Q(o)$ and satisfy $\mathbb{P}(\sigma_n = 1) = p$. Furthermore denote by $\pi_1 : \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\} \rightarrow \mathbb{T}^2$ respectively $\pi_{1,2} : \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\} \rightarrow \mathbb{T}^2 \times [0, \lambda]$ the projection onto the first coordinate respectively the first two coordinates. Then $Z = \pi_1(X) \subset \mathbb{T}^2$ is a homogeneous Poisson point process on \mathbb{T}^2 with intensity λ and $Y = \pi_{1,2}(X)$ is a homogeneous Poisson point process on $\mathbb{T}^2 \times [0, \lambda]$ with intensity 1. Note that to obtain from X a realization of the confetti process on \mathbb{T}^2 it suffices to independently simulate a further $\{\pm 1\}$ -marked homogeneous Poisson process $X' \subset \mathbb{T}^2 \times [\lambda, \infty) \times \{\pm 1\}$ and consider the coloring of \mathbb{T}^2 induced by the superposition of X and X' . Let us denote by $C_s^{(1)}$ the event $\mathbb{T}^2 \subset \bigcup_{z \in Z} Q_{1/2}(z)$. In this case the process X' is not needed to determine the coloring of \mathbb{T}^2 induced by $X \cup X'$ (more precisely, X' is still not needed after perturbing the squares in X by at most $1/4$ in each direction). Just as in Section 4 one proves that $C_s^{(1)}$ occurs whp.

Since the notation $\delta = \delta(s) \sim s^{-\gamma}$ is reserved for the mesh-size corresponding to the discretization we are about to construct, we use the notation $\delta_0 > 0$ for a temporary variable that could take the value of $\delta(s)$ or some related quantity. We want to formalize the notion of unstable configuration of squares whose connectivity properties can be altered by small perturbations in the space or time dimension. For $\delta_0 > 0$ and $z, z' \in Z$, $z \neq z'$ we say that $\{z, z'\}$ forms a *spatially δ_0 -unstable pair* if and only if $z - z' \in A_{\delta_0}$, where

$$A_{\delta_0} = \{z \in \mathbb{T}^2 : z \in Q_{2\delta_0}(o) \oplus (\{o, \pm e_2\} \oplus [-2e_1, 2e_1] \cup (\{o, \pm e_1\} \oplus [-2e_2, 2e_2]))\},$$

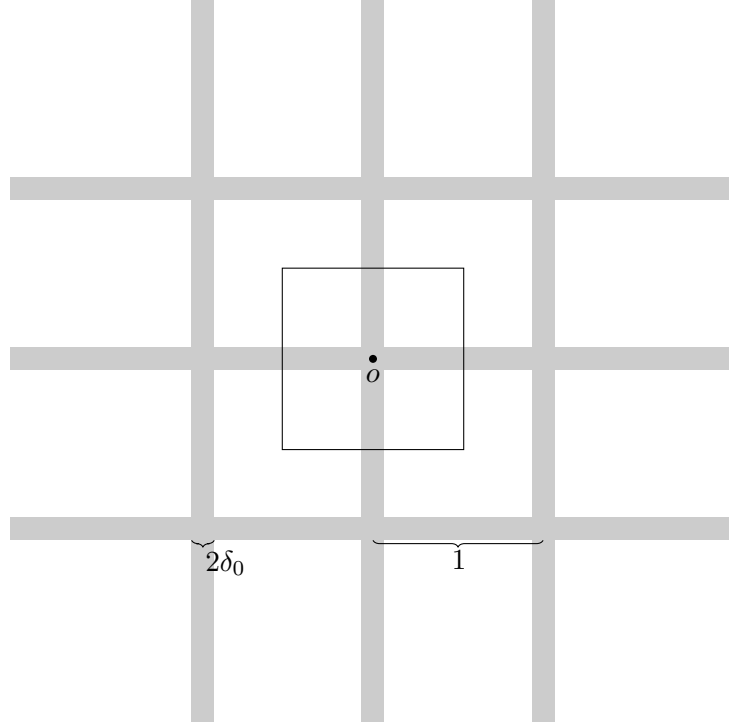


Figure 1: the square $Q(o)$ (black) and the set A_{δ_0} (gray)

compare also Figure 1. Similarly, we can define instabilities with respect to time. For $t, t' > 0$, $t \neq t'$, we say that $\{t, t'\}$ forms a *temporally δ_0 -unstable pair* if $|t - t'| < \delta_0$.

Finally for $y = (z, t)$, $y' = (z', t') \in \mathbb{T}^2 \times [0, \lambda]$ we say that $\{y, y'\}$ forms a *δ_0 -unstable pair* if the following two properties hold

- (i) $|y - y'|_\infty \leq 2 + \delta_0$
- (ii) $\{z, z'\}$ forms a spatially δ_0 -unstable pair or $\{t, t'\}$ forms a temporally δ_0 -unstable pair.

As already mentioned above, due to the coarseness of the discretization we *cannot* exclude the existence of δ -unstable pairs whp. Let us first note that the total number of δ -unstable pairs in a $\log s$ -square is bounded from above by a constant whp.

Lemma 5. *Let X be as above, let $\gamma_0 > 0$ be arbitrary and let $\delta_0 = \delta_0(s) \leq s^{-\gamma_0}$. Then for all $\gamma_0 > 0$ there exists $K > 0$ such that the probability that the total number of δ_0 -unstable pairs contained in $Q_{\log s}(o) \times [0, \lambda]$ is larger than K is of order $O(s^{-3})$.*

Proof. By the Poisson concentration inequality (see e.g. [14, Lemma 1.2]) there exist constants $c_1, K_1 > 0$ such that for $M = K_1 (\log s)^3$ we have $\mathbb{P}(Z(Q_{\log s}(o)) \geq M) \leq c_1 s^{-3}$. Let y_1, \dots, y_M be iid points placed uniformly in $Q_{\log s}(o) \times [0, \lambda]$. Next denote by A' the event describing the existence of $1 \leq j \leq M$ such that y_j is contained in more than K_2 pairs which are δ_0 -unstable, where $K_2 > 0$ is a constant to be specified. Then it is easy to show that we can choose $K_2 > 0$ with $\mathbb{P}(A') \leq c_2 s^{-3}$ for some constant $c_2 > 0$. Furthermore we claim the existence of $c_3, K_3 > 0$ (independent of s) such that the probability that the number of δ_0 -unstable pairs formed by the points y_1, \dots, y_M placed uniformly in $Q_{\log s}(o) \times [0, \lambda]$ exceeds K_3 is bounded from above by $c_3 s^{-3}$. Indeed, suppose that for $i \geq 1$ the points y_1, \dots, y_i are already placed and denote by A_i the event that y_{i+1} forms a δ_0 -unstable pair with one of y_1, \dots, y_i . Then the conditional

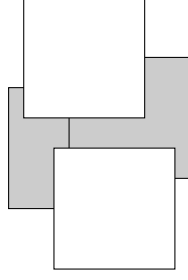


Figure 2: configuration sensitive to small perturbation of white squares

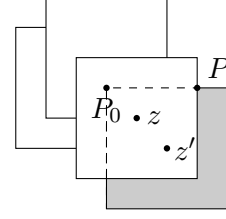


Figure 3: y is boundary-visible from y'

probability of A_i given the location of the points y_1, \dots, y_i can be bounded from above by $c'_4 M \delta_0 \leq c_4 (\log s)^3 \delta_0$ for suitable constants $c_4, c'_4 > 0$. Now observe that if A' does not occur and there exist more than K_4 pairs which are δ_0 -unstable, then the event A_i occurs for at least K_3/K_2 values of i . Thus the probability of obtaining more than K_3 unstable pairs is bounded from above by $c_1 s^{-3} + M^{K_3/K_2} c_4^{K_3/K_2} (\log s)^{3K_3/K_2} \delta_0^{K_3/K_2}$, so that it suffices to choose any K_3 with $\gamma_0 K_3/K_2 > 3$. \square

It is easy to visualize that a small perturbation of squares centered at the nodes of a δ_0 -unstable pair could also change the connectivity properties of neighboring squares. For instance, a small perturbation of the (spatially δ -unstable) white squares in Figure 2 could lead to the disconnectedness of the two black squares (even if these are not spatially δ -unstable). A naive approach would be to say $\{y, y', y''\}$ forms a δ_0 -unstable triple if $\{y, y'\}$ constitutes a δ_0 -unstable pair and $\min(|z - z''|_\infty, |z' - z''|_\infty) \leq 2$. However, as $\pi_1(Y) \subset \mathbb{T}^2$ forms a homogeneous Poisson process with intensity $\lambda = 50 \lfloor \log s \rfloor$, at least heuristically we would expect that for a fixed δ_0 -unstable pair there exist $\asymp \log s$ nodes y'' such that $\{y, y', y''\}$ forms a (naively) δ_0 -unstable triple. However, a direct adaptation of the coupling trick of [4] will not work if the number of unstable triples is that large. Therefore we take only those neighbors into account whose connectivity properties could be destroyed by a small perturbation of the unstable pair. Therefore we formalize the notion of leaves visible at locations close to the boundary of another leaf. Let $\varphi = \{x_n\}_{1 \leq n \leq N} = \{(z_n, t_n, \sigma_n)\}_{1 \leq n \leq N}$ and for $P \in \mathbb{T}^2$ define the set $\varphi_P = \{x_n \in \varphi : P \in Q(z_n) \setminus \partial Q(z_n)\}$. For $x = (z, t, \sigma)$, $x' = (z', t', \sigma') \in \varphi$ we say that x is *boundary-visible from x' in φ* (or also that $y = (z, t)$ is *boundary-visible from $y' = (z', t')$ in φ*) if there exists a corner P_0 of $\partial Q(z')$ and a point $P \in \partial Q(z') \cap \partial Q(z)$ such that $P_0 \in Q(z)$ and $t = \text{height}_{\varphi_{P_0}}(P)$ (see also Figure 3). First we note that for given x' the number of $x \in \varphi$ which are boundary-visible from x' is rather small.

Lemma 6. *Let $x' \in \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ be arbitrary and let X be as above. Then there exists a constant $c > 0$ such that*

$$\mathbb{P}(\#\{x \in X : x \text{ is boundary-visible from } x' \text{ in } X \cup \{x'\}\} \geq c \log s / \log \log s) \leq s^{-3}.$$

Proof. Without loss of generality, we may assume $z' = o$, so that $x' = (o, t')$. Furthermore it suffices to prove a corresponding bound for the number N_1 of $x \in X$ such that $P_0 = -e_1/2 - e_2/2 \in Q(z)$ and such that for $P = [P_0, P_0 + e_1] \cap \partial Q(z)$ we have $t = \text{height}_{\varphi_{P_0}}(P)$. Observe that for $n \geq 1$ if $N_1 \geq n$ then there exist $x_1, x_2, \dots, x_n \in X \cap Q_3(o) \times [0, \lambda]$ satisfying

- (i) $t_1 \leq t_2 \leq \dots \leq t_n$ and

$$(ii) \ z_1^{(1)} \leq z_2^{(1)} \leq \dots \leq z_n^{(1)},$$

where $z_i^{(1)}$ denotes the first coordinate of z_i . Therefore using the Slivnyak-Mecke formula we compute

$$\begin{aligned} \mathbb{P}(N_1 \geq n) &\leq 3^n \left(\int_{-3/2}^{3/2} \int_{-3/2}^{z_1^{(1)}} \dots \int_{-3/2}^{z_{n-1}^{(1)}} 1 dz_n^{(1)} \dots dz_1^{(1)} \right) \left(\int_0^\lambda \int_0^{t_1} \dots \int_0^{t_{n-1}} 1 dt_n \dots dt_1 \right) \\ &= 3^n \left(\int_0^3 \int_0^{\xi_1} \dots \int_0^{\xi_{n-1}} 1 d\xi_1 \dots d\xi_n \right) \frac{\lambda^n}{n!} \\ &= \frac{9^n \lambda^n}{n!^2}. \end{aligned}$$

Using Stirling's formula we conclude that there exists $c_1 > 0$ such that the latter expression is bounded from above by $c_1^n \lambda^n / n^{2n}$ for all sufficiently large n . In particular choosing $n = c \log s / \log \log s$ then for all s sufficiently large we compute

$$\begin{aligned} \mathbb{P}(N_1 \geq n) &\leq c_1^n \left(\frac{50 (\log \log s)^2}{c^2 \log s} \right)^n \\ &= \exp \left(n \left(-\log \log s + \log (50c_1/c^2) + 2 \log \log \log s \right) \right) \\ &\leq \exp \left(-\frac{n \log \log s}{2} \right) \\ &= \exp \left(-\frac{c \log s}{2} \right). \end{aligned}$$

In particular choosing $c = 6$ proves the claim. \square

For $y \in \pi_{1,2}(\varphi)$ define $\varphi_{\delta_0}^y$ to be the set of all y' in $\pi_{1,2}(\varphi)$ such that $\{y, y'\}$ does not form a δ_0 -unstable pair. We say that $\{x, x', x''\}$ forms a δ_0 -unstable triple (or also that $\{y, y', y''\}$ forms a δ_0 -unstable triple) if $\{y, y'\}$ forms a δ_0 -unstable pair and y'' is boundary-visible from y in $\varphi_{\delta_0}^y$. By a δ_0 -bad component of $\varphi = \{x_n\}_{1 \leq n \leq N}$ we denote a connected component of the graph with vertex set φ and where an edge is drawn between $x_1, x_2 \in \varphi$ if $\{x_1, x_2\}$ forms a δ_0 -unstable pair or if there exists a δ_0 -unstable triple containing x_1 and x_2 . As in [4, Lemma 6.4] we first prove that δ_0 -bad components are rather small (to be more precise, of sub-logarithmic size)

Lemma 7. *Let X be as above, let $\eta, \gamma_0 > 0$ be arbitrary and let $\delta_0 = \delta_0(s) \leq s^{-\gamma_0}$. For $\eta > 0$ denote by $C_{s, \gamma_0, \eta}^{(2)}$ the event that no δ_0 -bad component of X consists of more than $\eta \log s$ vertices. Then for all $\eta, \gamma_0 > 0$ the event $C_{s, \gamma_0, \eta}^{(2)}$ occurs whp.*

Proof. By Palm calculus it suffices to prove the existence of a constant $c_1 > 0$ such that for any fixed $(z_0, t_0, \sigma_0) \in \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ the probability that the δ_0 -bad component of $X \cup \{(z_0, t_0, \sigma_0)\}$ containing (z_0, t_0, σ_0) consists of more than $\eta \log s$ vertices is of order $O(s^{-5/2})$. Next observe that to prove this statement it suffices to consider the confetti process on a log s -square centered at z_0 . Indeed for all $\eta > 0$ sufficiently small if the δ_0 -bad component containing x consists of at most $\eta \log s$ vertices then its d_∞ -diameter is bounded from above by $\log s$. Also note that as before there exists constants $c_1, K_1 > 0$ such that for $M = K_1 (\log s)^3$ we have $\mathbb{P}(Z(Q_{\log s}(o)) \geq M) \leq c_1 s^{-3}$ and we denote by y_1, \dots, y_M iid points placed uniformly in $Q_{\log s}(o) \times [0, \lambda]$.

By Lemma 5 there exist constants $c_2, K > 0$ such that for all sufficiently large $s > 0$ the number of such δ_0 -unstable pairs in $\{y_1, \dots, y_M\}$ is bounded from above by some constant $K > 0$ with probability $\geq 1 - c_2 s^{-3}$. Furthermore by Lemma 6 there exist $c_3, c_4 > 0$ such that for all s sufficiently large the probability that for all $1 \leq i \leq M$ there exist at most $c_3 \log s / \log \log s$

elements y_j such that y_j is boundary-visible from y_i in $\{y_1, \dots, y_M\}$ is bounded from below by $1 - c_4 s^{-5/2}$. Combining the two observations completes the proof of the lemma. \square

Remark 1. The above argument shows that the δ_0 -bad components are of order $O(\log s / \log \log s)$. This approach was motivated by a suggestion of an anonymous referee. Originally she/he noted that although the number of leaves falling into the cube $Q(o) \times [0, \lambda]$ is $\asymp \log s$, the number of *visible* leaves intersecting $Q(o)$ (and not only its boundary) should be much smaller. For instance if one could show that the probability that more than k leaves are visible decays as k^{-ck} then we could deduce that whp only $O(\log s / \log \log s)$ are relevant. This tail behavior is rather easily established in dimension 1 due to the linear order of leaves. However it seems not completely trivial to prove the analog in dimension 2, since configuration of visible square-shaped leaves do not form a linear but a tree-like (or rather DAG-like) structure. In particular if the tree structure is very balanced and binary, naive estimates do not seem strong enough in order to obtain tail-estimates strictly stronger than exponential. As hinted by the referee if this estimate could be made rigorous, it would lead to further simplifications of the proof.

After these preparations let us recall the coupling trick introduced by Bollobás and Riordan in [4, Section 6.3]. Let $0 < p_1 < p_2 < 1$ and for $i \in \{1, 2\}$ denote by $X_i = \{(z_{i,n}, t_{i,n}, \sigma_{i,n})\}_{1 \leq n \leq N_i} \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ two $\{\pm 1\}$ -marked homogeneous Poisson processes with intensity 1 and $\mathbb{P}(\sigma_{i,n} = 1) = p_i$, where $0 < p_1 < p_2 < 1$. Write $\delta = \delta(s) = (4\lceil s^\gamma/4 \rceil)^{-1}$ for some $\gamma > 0$ to be specified in the proof of Theorem 12. Furthermore let $\delta_1 = (\lceil (\sqrt{\delta})^{-1} \rceil)^{-1}$ and let us write $R_1 = [0.25s, 8.75s] \times [-0.25s, 1.25s]$ and $R_2 = [0.5s, 8.5s] \times [-0.5s, 1.5s]$. For $\delta_0 > 0$ let us construct from $\varphi = \{(z_n, t_n, \sigma_n)\}_{1 \leq n \leq N} \subset \mathbb{T}^2 \times \mathbb{R} \times \{\pm 1\}$ a $\{\pm 1\}$ -marked set φ^{δ_0} with the property that black squares are delayed and shrinkened, while white ones are advanced and enlarged. More precisely write $\varphi^{\delta_0} = \{(z_n, t_n + \sigma_n \delta_0, \sigma_n)\}_{1 \leq n \leq N}$, where the leaf associated with $(z_n, t_n + \sigma_n \delta_0)$ is given by $Q_{1-2\sigma_n \delta_0}(o)$. Furthermore defining the coloring $\psi^{\delta_0} = \psi_{\varphi^{\delta_0}}$ we prove the following result.

Lemma 8. *There exists a coupling between X_1 and X_2 with $\mathbb{P}(E_g) \rightarrow 1$ as $s \rightarrow \infty$ where E_g denotes the intersection of the following events.*

- (i) $N_1 = N_2$ and $|y_{1,i} - y_{2,i}|_\infty \leq \delta_1$ for all $1 \leq i \leq N_1$
- (ii) for all $1 \leq i \leq N_1$ we have $\sigma_{2,i} \geq \sigma_{1,i}$
- (iii) if there exists a black horizontal crossing of R_1 in the dead leaves process induced by X_1 then there exists a $\psi^{2\delta}$ -black horizontal crossing of R_2 in the dead leaves process induced by X_2 .

Let us begin by describing the construction of this coupling. Define $K_1 = 100s^2 \lambda \delta_1^{-3}$ and subdivide $\mathbb{T}^2 \times [0, \lambda]$ into K_1 cubes Q_i of the form $Q_i = Q_{\delta_1}(z) \times [(k-1)\delta_1, k\delta_1]$ for some $z \in \delta_1 \mathbb{Z}^2$ and $1 \leq k \leq \lambda/\delta_1$. Denote by N a Poisson random variable with mean $100s^2 \lambda$ and independently for each $1 \leq j \leq N$ choose an index s_j uniformly from $\{1, \dots, K_1\}$. Then for $i \in \{1, 2\}$ the unmarked point process $Y_i = \{y_{i,1}, \dots, y_{i,N}\} \subset \mathbb{T}^2 \times [0, \lambda]$ corresponding to X_i is constructed by choosing $y_{i,j}$ uniformly from Q_{s_j} . Our task is to provide a suitable coupling of $(y_{1,j}, \sigma_{1,j})$ and $(y_{2,j}, \sigma_{2,j})$ for all $1 \leq j \leq N$.

Write $\delta_2 = \sqrt{\delta_1}$. For $a, b \in \{1, \dots, N\}$ we say that $\{a, b\}$ forms a *potentially δ_2 -unstable pair* if it is possible to find $\tilde{y}_a \in Q_{s_a}$ and $\tilde{y}_b \in Q_{s_b}$ such that $\{\tilde{y}_a, \tilde{y}_b\}$ forms a δ_2 -unstable pair (here we do not necessarily assume $\tilde{y}_a, \tilde{y}_b \in Y_1$ or $\tilde{y}_a, \tilde{y}_b \in Y_2$). Similarly for $a, b, c \in \{1, \dots, N\}$ we say that $\{a, b, c\}$ forms a *potentially δ_2 -unstable triple* if it is possible to find $\tilde{y}_j \in Q_{s_j}$, $1 \leq j \leq N$ such that $\{\tilde{y}_a, \tilde{y}_b, \tilde{y}_c\}$ forms a δ_2 -unstable triple in $\{\tilde{y}_j\}_{1 \leq j \leq N}$. Finally we say that $C \subset \{1, \dots, N\}$ forms a *potentially δ_2 -bad component* if C is a connected component in the graph with vertex set $\{1, \dots, N\}$ and where $1 \leq a, b \leq N$ are connected by an edge if $\{a, b\}$ forms a potentially δ_2 -unstable pair or if there exists a potentially δ_2 -unstable triple containing both a and b .

Lemma 9. For $\eta > 0$ denote by $C_{s,\eta}^{(3)}$ the event of the absence of potentially δ_2 -bad components consisting of more than $\eta \log s$ vertices. Then for all $\eta > 0$ the event $C_{s,\eta}^{(3)}$ occurs whp.

Proof. First observe that if $\{y_a, y_b\}$ forms a δ_2 -unstable pair, then for all $\tilde{y}_a \in Q_{s_a}$, $\tilde{y}_b \in Q_{s_b}$ the pair $\{\tilde{y}_a, \tilde{y}_b\}$ forms a $\delta_2 + 2\delta_1$ -unstable pair. Furthermore suppose that $\{y_a, y_b, y_c\}$ forms a δ_2 -unstable triple. In particular assume that y_c is boundary-visible from y_a in $(Y_1)_{\delta_2}^{y_a}$ and for $1 \leq j \leq N$ let $\tilde{y}_j \in Q_{s_j}$ be arbitrary. Then we claim \tilde{y}_a and \tilde{y}_c are contained in the same $\delta_2 + 2\delta_1$ -bad component with respect to $\{\tilde{y}_j\}_{1 \leq j \leq N}$. Without loss of generality we may assume that z_c lies to the north-west of z_a and that a point $P \in \partial Q(z_a) \cap \partial Q(z_c)$ satisfying the assumption in the definition of the boundary-visibility property lies on the upper horizontal boundary of $Q(z_a)$. Furthermore write $\tilde{P} = P + \pi_1(\tilde{z}_c - z_c)e_1 + \pi_2(\tilde{z}_a - z_a)e_2$. First we claim $|\tilde{z}_c - \tilde{P}|_\infty \leq 1/2$ (and similarly for \tilde{z}_a). Indeed, as $|\pi_1(\tilde{P} - \tilde{z}_c)| = 1/2$, we have

$$\begin{aligned} |\tilde{z}_c - \tilde{P}|_\infty > 1/2 &\iff |\pi_2(\tilde{P} - \tilde{z}_c)| > 1/2 \\ &\iff |\pi_2(P - z_c) + \pi_2(z_c - \tilde{z}_c) + \pi_2(\tilde{z}_a - z_a)| > 1/2. \end{aligned}$$

In particular $|\tilde{z}_c - \tilde{P}|_\infty > 1/2$ implies $1/2 - 2\delta_1 \leq |\pi_2(P - z_c)| \leq 1/2$ so that $\{z_a, z_c\}$ forms a spatially $2\delta_1$ -unstable pair, thereby contradicting $y_c \in (Y_1)_{\delta_2}^{y_a}$. Therefore, we may henceforth assume $|\tilde{z}_c - \tilde{P}|_\infty = |\tilde{z}_a - \tilde{P}|_\infty = 1/2$. Denote by \tilde{P}_0 the upper-left corner of $Q(\tilde{z}_a)$. Under these assumptions, it follows that if $\{\tilde{y}_c, \tilde{y}_a\}$ does not form a $\delta_2 + 2\delta_1$ -unstable pair then there exists $1 \leq d \leq N$, $d \neq a$ with

- (i) $\tilde{y}_d \in (\{\tilde{y}_j\}_{1 \leq j \leq N})_{\delta_2 + 2\delta_1}^{\tilde{y}_a}$,
- (ii) $|\tilde{y}_d - \tilde{P}_0|_\infty \leq 1/2$,
- (iii) $\tilde{t}_d \leq \tilde{t}_c$ and
- (iv) $\pi_1(\tilde{y}_d) \geq \pi_1(\tilde{y}_c)$.

Among all these values choose d such that \tilde{t}_d is minimal. We are finished if $d = c$, so we may assume from now that $d \neq c$. In particular \tilde{y}_d is boundary-visible from \tilde{y}_a so that \tilde{y}_d, \tilde{y}_a are contained in the same $\delta_2 + 2\delta_1$ -bad component. Therefore we are finished if $\{\tilde{y}_c, \tilde{y}_d\}$ forms a $2\delta_1$ -unstable pair, so that we may assume $\tilde{t}_d + 2\delta_1 < \tilde{t}_c$ and $\pi_1(\tilde{y}_d) \geq \pi_1(\tilde{y}_c) + 2\delta_1$. But then we have $\pi_1(y_d) \geq \pi_1(y_c)$ and $t_d \leq t_c$ contradicting the assumption that $\{y_a, y_b, y_c\}$ forms a δ_2 -unstable triple. Using Lemma 7, we thus see that with high probability no potentially δ_2 -bad component contains more than $\eta \log s$ vertices. \square

If C is a potentially δ_2 -bad component then we say $B(C)$ occurs for $\{(y_{1,j}, \sigma_{1,j})\}_{1 \leq j \leq N}$ if there exist $a, b \in C$ such that $\{y_{1,a}, y_{1,b}\}$ forms a 16δ -unstable pair. To prove Lemma 8 we need the following probabilistic result.

Lemma 10. There exists a coupling between $\{(y_{1,j}, \sigma_{1,j})\}_{1 \leq j \leq N}$ and $\{(y_{2,j}, \sigma_{2,j})\}_{1 \leq j \leq N}$ such that the probability of occurrence of the following events tends to 1 as $s \rightarrow \infty$.

- (i) $|y_{1,i} - y_{2,i}|_\infty \leq \delta_1$ for all $1 \leq i \leq N$
- (ii) for all $1 \leq i \leq N$ we have $\sigma_{2,i} \geq \sigma_{1,i}$
- (iii) if $C \subset \{1, \dots, N\}$ is a potentially δ_2 -bad component and if there exist $a, b \in C$ with $\sigma_{1,a} = 1$ and $\sigma_{2,b} = -1$ then $B(C)$ does not occur and we have $y_{1,j} = y_{2,j}$ for all $j \in C$

Proof. Choose $\eta > 0$ such that $2s^{-\gamma/3} \leq s^{\log(p_2 - p_1)\eta}$ holds for all sufficiently large $s > 0$ and assume that $C_{s,\eta}^{(3)}$ holds (recall from Lemma 9 that this happens whp). To construct the desired coupling between $\{(y_{1,j}, \sigma_{1,j})\}_{1 \leq j \leq N}$ and $\{(y_{2,j}, \sigma_{2,j})\}_{1 \leq j \leq N}$ we first define a preliminary

version that may be considered as *natural coupling*. This natural coupling is constructed simply by choosing $y_{1,j} = y_{2,j}$ to be uniformly distributed in Q_{s_j} . Furthermore we choose $\sigma_{1,j} = 1$ with probability p_1 and the value of $\sigma_{2,j}$ is determined conditionally on the value of $\sigma_{1,j}$. If $\sigma_{1,j} = 1$ then we also put $\sigma_{2,j} = 1$, but if $\sigma_{1,j} = -1$ then we put $\sigma_{2,j} = 1$ with probability $(p_2 - p_1)/(1 - p_1)$. Starting from this simple coupling we construct the final coupling on each of the potentially δ_2 -bad components separately.

Let $C \subset \{1, \dots, N\}$ be a potentially δ_2 -bad component. It is easy to check that there exists a constant $c_1 > 0$ such that for all $1 \leq j \leq K_1$ and all $\tilde{y} \in Q_j$ we have $\nu_3((\tilde{y} + F_{16\delta}) \cap Q_j) \leq c_1 \delta_1^2 \delta$, where $F_{16\delta}$ denotes the set of all $(z, t) \in \mathbb{T}^2 \times \mathbb{R}$ such that $\{(z, t), (o, 0)\}$ forms a 16δ -unstable pair. As C consists of at most $\eta \log s$ vertices, we see that the expected number of 16δ -unstable pairs in C is bounded from above by $\eta \log s \cdot c_1 \delta_1^2 \delta / \delta_1^3 \leq s^{-\gamma/3}$ for all sufficiently large values of s .

On the other hand, denote by $G(C)$ the event that we have $\sigma_{2,j} = 1$ and $\sigma_{1,j} = -1$ for all $j \in C$, so that

$$\begin{aligned} \mathbb{P}(G(C)) &\geq (p_2 - p_1)^{\eta \log s} \\ &= s^{\eta \log(p_2 - p_1)} \\ &\geq 2s^{-\gamma/3}. \end{aligned}$$

In particular we obtain $\mathbb{P}(G(C) \setminus B(C)) \geq \mathbb{P}(B(C))$. The final coupling is now constructed by using the cross-over coupling described in [4]. For the convenience of the reader we briefly recall this technique.

Let X_1^*, X_2^* be random variables with marginals distributed as X_1, X_2 and that are coupled according to the natural coupling described above. Choose $G'(C) \subset G(C) \setminus B(C)$ with $\mathbb{P}(G'(C)) = \mathbb{P}(B(C))$ and a measure-preserving bijection f on our probability space that maps $B(C) \cup G'(C)$ to itself and where $B(C)$ is mapped into $G'(C)$ and vice versa. Then we put $X_1 = X_1^*$ and define X_2 by

$$X_2(\omega) = \begin{cases} X_2^*(\omega) & \text{if } \omega \notin B(C) \cup G'(C) \\ X_2^*(f(\omega)) & \text{if } \omega \in B(C) \cup G'(C) \end{cases}.$$

In particular the final coupling has the properties

- (i) If $\omega \notin B(C) \cup G'(C)$, then $y_{1,j} = y_{2,j}$ for all $j \in C$ and there do not exist 16δ -unstable pairs in C .
- (ii) If $\omega \in B(C)$ then $\sigma_{2,j} = 1$ for all $j \in C$.
- (iii) If $\omega \in G'(C)$ then $\sigma_{1,j} = -1$ for all $j \in C$.

□

The second step in the proof of Lemma 8 is to show that given a coupling as described in Lemma 10, the occurrence of a ψ_{X_1} -black horizontal crossing of R_1 implies the existence of a $\psi^{2\delta}$ -black horizontal crossing of R_2 in the coloring associated with X_2 . After having constructed the explicit coupling this is a completely elementary geometric (i.e. deterministic) problem. Unfortunately its proof is nevertheless rather intricate so that we postpone it to Section 6. We now explain how Lemma 8 and the Friedgut-Kalai sharp-threshold result can be used to obtain a proof of $\theta(p) > 0$ for $p > 1/2$.

Let us begin by recalling the following variant (which appears in [8, Lemma 1]) of a sharp-threshold result due to Friedgut and Kalai (see [9, Theorem 2.1]). For $m, n \geq 1$ we say that an event $E \subset \{0, \pm 1\}^n$ has symmetry of order m if it is invariant under the action of a subgroup $\Gamma \subset \Sigma_n$ with the property that all of its orbits consist of at least m elements (here Σ_n denotes the symmetric group on n symbols). Furthermore for $0 < p_b, p_w < 1$ with $p_w + p_b < 1$ we

write \mathbb{P}_{p_b, p_w} for the product measure on the space $(\{0, \pm 1\}^n, \mathcal{P}(\{0, \pm 1\}^n))$ determined by the marginals $\mathbb{P}_{p_b, p_w}(\omega \in \{1\} \times \{0, \pm 1\}^{n-1}) = p_b$ and $\mathbb{P}_{p_b, p_w}(\omega \in \{-1\} \times \{0, \pm 1\}^{n-1}) = p_w$.

Proposition 11. *There is an absolute constant $c_3 > 0$ such that if $0 < q_w < p_w < 1/e$ and $0 < p_b < q_b < 1/e$ if $E \subset \{0, \pm 1\}^n$ is increasing and has symmetry of order m and if $\mathbb{P}_{p_b, p_w}(E) > \eta$, then $\mathbb{P}_{q_b, q_w}(E) > 1 - \eta$ whenever*

$$\min\{q_b - p_b, p_w - q_w\} \geq c_3 \log(1/\eta) p_{\max} \log(1/p_{\max}) / \log m,$$

where $p_{\max} = \max\{q_b, p_w\}$.

We apply this result in the following situation. Let $X = (z_n, t_n, \sigma_n)_{1 \leq n \leq N}$ be an independently $\{\pm 1\}$ -marked homogeneous Poisson point process on $\mathbb{T}^2 \times [0, \lambda]$ with intensity 1 with $\mathbb{P}(\sigma_n = 1) = p$. Furthermore write $Y_b = \{(z_n, t_n) : \sigma_n = 1\}_{(z_n, t_n, \sigma_n) \in X}$ and $Y_w = \{(z_n, t_n) : \sigma_n = -1\}_{(z_n, t_n, \sigma_n) \in X}$. Discretize $\mathbb{T}^2 \times [0, \lambda]$ into $n_0 = n_0(s) = 100s^2\lambda\delta^{-3}$ cubes Q_i of the form $Q_i = Q_\delta(z) \times [(k-1)\delta, k\delta]$ where $z \in \mathbb{T}^2 \cap \delta\mathbb{Z}^2$ and $1 \leq k \leq \lambda\delta^{-1}$. We call Q_i *white* if $Q_i \cap Y_w \neq \emptyset$ and *black* if $Q_i \cap Y_b \neq \emptyset$ and $Q_i \cap Y_w = \emptyset$. Otherwise (i.e. if $Q_i \cap Y = \emptyset$) we say that Q_i is *neutral*. Then the states of different cubes are independent and the probabilities that a fixed cube has a specified state are given by

$$\begin{aligned} p_{\text{white}} &= 1 - \exp(\delta^3(1 - p)) \\ p_{\text{black}} &= \exp(-\delta^3(1 - p))(1 - \exp(-\delta^3 p)) \\ p_{\text{neutral}} &= \exp(-\delta^3). \end{aligned}$$

In other words, this probability measure is a product measure on $\{0, \pm 1\}^{n_0}$ and our goal is to apply Proposition 11. The relevant subgroup $\Gamma \subset \Sigma_{n_0}$ is generated by the translations $(x, y, z) \mapsto (x + \delta, y, z)$ and $(x, y, z) \mapsto (x, y + \delta, z)$ of $(\mathbb{T}^2 \cap \delta\mathbb{Z}^2) \times ((0, \lambda] \cap \delta\mathbb{Z})$.

We claim that if $\omega \in \{0, \pm 1\}^{n_0}$ is a configuration that results from a realization $\varphi = \{(z_n, t_n, \sigma_n)\}_{1 \leq n \leq N}$ of X having the property that R_2 has a $\psi^{2\delta}$ -black horizontal crossing, then there is a $\psi_{\varphi'}$ -black horizontal crossing of R_2 for *any* $\varphi' \subset \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$ that induces the same discrete configuration $\omega \in \{0, \pm 1\}^{n_0}$ as φ . This may be seen as follows. Any discrete configuration $\omega \in \{0, \pm 1\}^{n_0}$ determines a coloring ψ_ω . Namely define $\varphi_\omega = \{(z, t + (\omega(z, t) - 1)\delta/2, \omega(z, t))\}_{(z, t) \in (\delta\mathbb{Z}^2 \cap \mathbb{T}^2) \times (\delta\mathbb{Z} \cap (0, \lambda]), \omega(z, t) \in \{\pm 1\}}$, where the mark of $(z, t + (\omega(z, t) - 1)\delta/2, \omega(z, t))$ is given by $Q_{1-2\omega(z, t)\delta}(o)$ (and also put $\psi_\omega = \psi_{\varphi_\omega}$). Now observe that ψ' black-dominates ψ_ω and that ψ_ω black-dominates $\psi_{\varphi^{2\delta}}$. Therefore the existence of a $\psi^{2\delta}$ -black horizontal crossing of R with respect to the coloring defined by φ implies the existence of a black horizontal crossing of R in the coloring defined by φ' . After these preliminaries the proof of $\theta(p) > 0$ for $p > 1/2$ is now completely analogous to [5, Theorem 17] and [4, Theorem 1.1] and is therefore omitted.

Theorem 12. *Let $p > 1/2$ be arbitrary. Then $\theta(p) > 0$.*

6 Proof of Lemma 8

In this section we provide a proof for the elementary geometric claim in Lemma 8 that was postponed in Section 5. The situation is as follows. For $i \in \{1, 2\}$ we are given finite sets $\{(y_{i,n}, \sigma_{i,n})\}_{1 \leq n \leq N} \in \mathbb{T}^2 \times [0, \lambda] \times \{\pm 1\}$.

- (i) $|y_{1,j} - y_{2,j}|_\infty \leq \delta_1$ for all $1 \leq j \leq N$
- (ii) $\sigma_{1,j} \leq \sigma_{2,j}$ for all $1 \leq j \leq N$
- (iii) for all potentially δ_2 -bad components $C \subset \{1, \dots, N\}$ the existence of $a, b \in C$ with $\sigma_{1,a} = 1$ and $\sigma_{2,b} = -1$ implies that $B(C)$ does not occur and that $y_{1,j} = y_{2,j}$ holds for all $j \in C$

Furthermore, without loss of generality, we may assume $C_s^{(1)}$. Our goal is to prove that these properties imply that if there exists a black horizontal crossing of $R_1 = [0.25s', 8.75s'] \times [-0.25s', 1.25s']$ in the confetti process induced by $\varphi_1 = \{(y_{1,n}, \sigma_{1,n})\}_{1 \leq n \leq N}$ then there exists a $\psi^{2\delta}$ -black horizontal crossing of $R_2 = [0.5s', 8.5s'] \times [-0.5s', 1.5s']$ in the confetti process induced by $\varphi_2 = \{(y_{2,n}, \sigma_{2,n})\}_{1 \leq n \leq N}$.

For $C \subset \{1, 2, \dots, N\}$ let us denote by $D(C)$ the event describing the existence of $n_1, n_2 \in C$ with $\sigma_{1,n_1} = 1$ and $\sigma_{2,n_2} = -1$. Define a sequence $\Delta = \{\delta^{(n)}\}_{1 \leq n \leq N}$ by $\delta^{(n)} = \delta_1 + 4\delta$ if n is contained in a potentially δ_2 -black component C such that $D(C)$ does not occur and put $\delta^{(n)} = 4\delta$ otherwise. By Lemma 10 the occurrence of $D(C)$ implies the non-existence of $m_1, m_2 \in C$ such that $\{y_{1,m_1}, y_{1,m_2}\}$ forms a spatially 16δ -unstable pair.

Lemma 13. *Let $1 \leq n_1, n_2 \leq N$ and suppose that $\{z_{1,n_1}, z_{1,n_2}\}$ forms a spatially $2\delta^{(n_1)} + 2\delta^{(n_2)}$ -unstable pair. Denote by C the potentially δ_2 -bad component containing n_1 and n_2 . Then $D(C)$ does not occur. In particular we have $\sigma_{1,n_1} = \sigma_{2,n_2}$.*

Proof. Assume the contrary, i.e. that $D(C)$ occurs. Then we have $\delta^{(n_1)} = \delta^{(n_2)} = 4\delta$ so that $\{z_{1,n_1}, z_{1,n_2}\}$ forms a spatially 16δ -unstable pair contradicting the occurrence of $D(C)$. \square

Lemma 14. *Let $1 \leq m, n \leq N$ be such that $|z_{1,n} - z_{1,m}|_\infty \leq 2$ and $t_{1,m} - \delta^{(m)} < t_{1,n} + \delta^{(n)}$. Denote by $\tilde{C} \subset \{1, \dots, N\}$ the union of the potentially δ_2 -bad connected components containing n and m . Furthermore assume the existence of $n', m' \in \tilde{C}$ with $\sigma_{1,m'} = 1$ and $\sigma_{2,n'} = -1$. Then we have $t_{1,m} < t_{1,n}$.*

Proof. Assume the contrary. Then $|t_{1,m} - t_{1,n}| \leq \delta^{(m)} + \delta^{(n)} \leq \delta_2$ so that m and n are contained in the same potentially δ_2 -bad component $C \subset \{1, \dots, N\}$. In particular the existence of $m', n' \in C$ with $\sigma_{1,m'} = 1$ and $\sigma_{2,n'} = -1$ implies $D(C)$. But then we have $\delta^{(n)} = \delta^{(m)} = 4\delta$ so that $\{t_{1,n}, t_{1,m}\}$ forms a temporally 8δ -unstable pair. This contradicts the discussion preceding Lemma 13. \square

To obtain a $\psi_{\varphi_2^{2\delta}}$ -black horizontal crossing of R_2 we construct a further coloring ψ^Δ that is black dominated by $\psi_{\varphi_2^{2\delta}}$ and show the existence of a black horizontal crossing of R_2 in this new coloring. To be more precise, let us define $\varphi^\Delta = \{\tilde{x}_n\}_{1 \leq n \leq N} = \{(z_{1,n}, t_{1,n} + \delta^{(n)}\sigma_{2,n}, \sigma_{2,n})\}_{1 \leq n \leq N}$, where the mark of $(z_{1,n}, t_{1,n} + \delta^{(n)}\sigma_{2,n}, \sigma_{2,n})$ is given by $Q_{1-2\sigma_{2,n}\delta^{(n)}}(o)$ (and also put $\psi^\Delta = \psi_{\varphi^\Delta}$). Observe that the coloring $\psi_{\varphi_2^{2\delta}}$ black-dominates the coloring ψ_{φ^Δ} so that it suffices to prove the existence of a ψ^Δ -black horizontal crossing of R_2 . Furthermore for $1 \leq n \leq N$ we write $\varphi_1(n) = \{x_{1,n}\} \cup \{x_{1,m} \in \varphi_1 : \sigma_{2,m} = -1, t_m < t_n\}$ and similarly $\varphi^\Delta(n) = \{\tilde{x}_n\} \cup \{\tilde{x}_m \in \varphi^\Delta : \sigma_{2,m} = -1, t_m - \delta^{(m)} < t_n + \delta^{(n)}\}$. Next we need a result to create $\varphi^\Delta(n)$ -black paths from $\varphi(n)$ -black paths.

Lemma 15. *Let $1 \leq n \leq N$ with $\sigma_{1,n} = 1$ and let $P_1, P_2 \in Q_{1-2\delta^{(n)}}(z_{1,n})$ with $\text{height}_{\varphi^\Delta(n)}(\{P_1, P_2\}) = \{t_{1,n} + \delta^{(n)}\}$ (where for a subset $S \subset \mathbb{R}^2$ we write $\text{height}_\varphi(S) = \{\text{height}(x) : x \in S\}$). Assume the existence of $P'_1, P'_2 \in Q(z_{1,n})$, $1 \leq m_1, m_2 \leq N$ and $i_1, i_2 \in \{1, 2\}$ such that the following items hold for all $k \in \{1, 2\}$*

- (i) $|\pi_{i_k}(P'_k) - \pi_{i_k}(z_{1,m_k})| = 1/2$
- (ii) $P'_k \in \partial Q(z_{1,m_k})$
- (iii) $|\pi_{i_k}(P'_k) - \pi_{i_k}(P_k)| \leq \delta^{(m_k)}$
- (iv) $\sigma_{1,m_k} = 1$

Assume furthermore the existence of a $\psi_{\varphi_1(n)}$ -black path $\gamma : [0, 1] \rightarrow \mathbb{T}^2$ with $\gamma(0) = P'_1$, $\gamma(1) = P'_2$. Then there exists a $\psi_{\varphi^\Delta(n)}$ -black path $\tilde{\gamma} : [0, 1] \rightarrow \mathbb{T}^2$ with $\tilde{\gamma}(0) = P_1$ and $\tilde{\gamma}(1) = P_2$.

Proof. Without loss of generality we may assume that P_1 lies to the north-west of P_2 . Assume the assertion of the lemma was wrong. This implies the existence of $1 \leq m_e, m_s \leq N$ with

- (i) $\sigma_{2,m_e} = \sigma_{2,m_s} = -1$
- (ii) $\max(t_{1,m_e} - \delta^{(m_e)}, t_{1,m_s} - \delta^{(m_s)}) < t_{1,n} + \delta^{(n)}$
- (iii) $\pi_1(z_{1,m_e}) - (\delta^{(m_e)} + 1/2) \in (\pi_1(P_1), \pi_1(P_2))$
- (iv) $\pi_2(z_{1,m_e}) - (\delta^{(m_e)} + 1/2) \in (\pi_2(P_2), \pi_2(P_1))$
- (v) $\pi_1(z_{1,m_s}) + (\delta^{(m_s)} + 1/2) \in (\pi_1(P_1), \pi_1(P_2))$
- (vi) $\pi_2(z_{1,m_s}) + (\delta^{(m_s)} + 1/2) \in (\pi_2(P_2), \pi_2(P_1))$
- (vii) $|z_{1,m_e} - z_{1,m_s}|_\infty < 1 + \delta^{(m_e)} + \delta^{(m_s)}$

(see also Figure 4 after replacing $Q(z_{1,m_e})$, $Q(z_{1,m_s})$ by $Q_{1+2\delta^{(m_e)}}(z_{1,m_e})$, $Q_{1+2\delta^{(m_s)}}(z_{1,m_s})$). Observe that by Lemma 14 we have $\max(t_{1,m_e}, t_{1,m_s}) < t_{1,n}$. We distinguish two cases.

First suppose $|z_{1,m_s} - z_{1,m_e}|_\infty < 1$. We claim that P_1 and P'_1 are contained in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$ (observe that this set consists of precisely two connected components since we assumed $|z_{1,m_s} - z_{1,m_e}|_\infty < 1$). Let us assume the contrary and furthermore – without loss of generality – that $i_1 = 1$. As P'_1 lies in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$ as P_2 we have

$$\begin{aligned} \pi_1(P'_1) - (\pi_1(z_{1,m_e}) - 1/2) &\geq \pi_1(P'_1) - (\pi_1(z_{1,m_s}) + 1/2) \\ &\geq 0. \end{aligned}$$

and also

$$\begin{aligned} \pi_1(P'_1) - (\pi_1(z_{1,m_e}) - 1/2) &= \pi_1(P'_1) - \pi_1(P_1) + \pi_1(P_1) - (\pi_1(z_{1,m_e}) - 1/2) \\ &\leq \delta^{(m_1)} + 0 \end{aligned}$$

In particular $\{z_{1,m_1}, z_{1,m_e}\}$ form a spatially $\delta^{(m_1)}$ -unstable pair, thereby contradicting Lemma 13. Therefore P_1 and P'_1 lie in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$. Of course, the same is true for P_2 and P'_2 . But this yields a contradiction to the existence of a $\psi_{\varphi_1(n)}$ -black path connecting P'_1 and P'_2 .

The reasoning in the case $|z_{1,m_s} - z_{1,m_e}|_\infty \geq 1$ is very similar, but we provide the details for the convenience of the reader. Denote by P_0 the south-west corner of $Q(z_{1,m_e})$. Then there exists a unique $1 \leq m_0 \leq N$, such that $t_{m_0} = \text{height}_{(\varphi_1)_{P_0} \cap (\varphi_1)_{\delta_2}^{y_{1,m_e}}}(P_0)$ (in particular y_{m_0} is boundary-visible from y_{m_e}). For an illustration of the situation see Figure 4. By Lemma 13 we may assume $\sigma_{1,m_0} = -1$. As $\{z_{1,m_0}, z_{1,m_e}\}$ does not form a spatially δ_2 -unstable pair we also see that P_1 and P_2 lie in different connected components of $Q(z_{1,n}) \setminus (Q(z_{1,m_0}) \cup Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$. Next we claim that P_1 and P'_1 are contained in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_0}) \cup Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$. Assume again the contrary and without loss of generality that $i_1 = 1$. As P'_1 lies in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_0}) \cup Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$ as P_2 we have again

$$\begin{aligned} \pi_1(P'_1) - (\pi_1(z_{1,m_e}) - 1/2) &\geq \pi_1(P'_1) - (\pi_1(z_{1,m_s}) + 1/2) - \delta^{(m_e)} - \delta^{(m_s)} \\ &\geq -\delta^{(m_e)} - \delta^{(m_s)}. \end{aligned}$$

and also

$$\begin{aligned} \pi_1(P'_1) - (\pi_1(z_{1,m_e}) - 1/2) &= \pi_1(P'_1) - \pi_1(P_1) + \pi_1(P_1) - (\pi_1(z_{1,m_e}) - 1/2) \\ &\leq \delta^{(m_1)} + 0 \end{aligned}$$

As we have $\delta^{(m_e)} = \delta^{(m_s)}$ we conclude that $\{z_{1,m_e}, z_{1,m_1}\}$ forms a spatially $2\delta^{(m_1)} + 2\delta^{(m_e)}$ -unstable pair, thereby contradicting Lemma 13.

Therefore P_1 and P'_1 lie in the same connected component of $Q(z_{1,n}) \setminus (Q(z_{1,m_0}) \cup Q(z_{1,m_e}) \cup Q(z_{1,m_s}))$. Of course, the same is true for P_2 and P'_2 . But this yields a contradiction to the existence of a $\psi_{\varphi_1(n)}$ -black path connecting P'_1 and P'_2 . \square

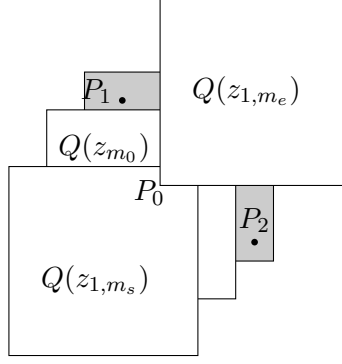


Figure 4: configuration in the proof of Lemma 15

We finally need a result allowing us to glue together paths obtained from Lemma 15.

Lemma 16. *Let $1 \leq n_1, n_2 \leq N$ be such that $\sigma_{1, n_1} = \sigma_{1, n_2} = 1$ and $t_{1, n_1} < t_{1, n_2}$. Furthermore let $P \in \mathbb{R}^2$ with $\text{height}_{\varphi_1}(P) = t_1$ and $\text{height}_{\varphi_1 \setminus \{x_{1, n_1}\}}(P) = t_2$. Then there exist $P_i \in \mathbb{R}^2$, $i \in \{1, 2\}$ with the following properties.*

- (i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1, n_i})$ for all $i \in \{1, 2\}$
- (ii) P_i is $\psi^\Delta(n_i)$ -black for all $i \in \{1, 2\}$
- (iii) let $k \in \{1, 2\}$ be such that $|\pi_k(P) - \pi_k(z_{1, n_1})| = 1/2$. Then $|\pi_k(P_i) - \pi_k(P)| \leq \delta^{(n_1)}$ holds for all $i \in \{1, 2\}$
- (iv) the linear segment $[P_1, P_2]$ is ψ^Δ -black

Proof. Without loss of generality we may assume that z_{1, n_2} lies to the south-east of z_{1, n_1} and that P lies on a right vertical boundary of $Q(z_{1, n_1})$. We distinguish 5 cases.

case 1a. Put $P_0 = z_{1, n_1} + e_1/2 - e_2/2$. First assume that $\{z_{1, n_1}, z_{1, n_2}\}$ forms a spatially $\delta^{(n_1)} + \delta^{(n_2)}$ -unstable pair. Assume additionally that there does *not* exist $1 \leq n_3 \leq N$ with $t_{1, n_3} < t_{1, n_2}$, $|\pi_1(P) - \pi_1(z_{1, n_3})| \leq 1/2$, $\pi_2(z_{1, n_3}) + 1/2 \in (\pi_2(z_{1, n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1, n_1}, z_{1, n_3}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_3)}$ -unstable pair (observe that the non-existence of n_3 is always satisfied if $\pi_2(z_{1, n_1}) > \pi_2(z_{1, n_2}) + 1 - \delta^{(n_1)} - \delta^{(n_2)}$). In particular (due to $C_s^{(1)}$) there exists $1 \leq n_4 \leq N$ with $t_{1, n_4} = \text{height}_{(\varphi_1)_{P_0} \cap (\varphi_1)_{\delta_2}^{y_{1, n_1}}}(P_0)$ (in particular y_{1, n_4} is boundary-visible from y_{1, n_1}). Observe that $\{z_{1, n_1}, z_{1, n_2}, z_{1, n_4}\}$ are contained in the same potentially δ_2 -bad connected component. In particular we conclude from Lemma 13 that $\sigma_{2, n_4} = 1$. Define $\xi_1 = (1 - 2 \cdot 1_{\pi_1(P) > \pi_1(z_{1, n_2})})$ and $\xi_2 = (1 - 2 \cdot 1_{\pi_2(z_{1, n_1}) - 1/2 > \pi_2(z_{1, n_2})})$. Then let us define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1, n_1}) - 1/2 + \delta^{(n_1)})e_2$ and $P_2 = (\pi_1(P) + \xi_1\delta^{(n_1)})e_1 + (\pi_2(z_{1, n_1}) - 1/2 + \xi_2\delta^{(n_1)})e_2$. See also Figure 5 for an illustration.

- (i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1, n_i})$ is clear.
- (ii) To show that P_i is $\psi^\Delta(n_i)$ -black assume for the sake of contradiction the existence of $1 \leq m \leq N$ with $t_{1, m} - \delta^{(m)} < t_{1, n_i} + \delta^{(n_i)}$, $\sigma_{2, m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1, m})$. Observe that by Lemma 14 we have $t_{1, m} < t_{1, n_i}$. By Lemma 13 and the non-existence of n_3 , we see that this is impossible.
- (iii) $|\pi_1(P) - \pi_1(P_i)| \leq \delta^{(n_1)}$ is clear.
- (iv) Suppose we could find $P' \in [P_1, P_2]$ and $1 \leq m \leq N$ with $t_{1, m} - \delta^{(m)} < t_{1, n_4} + \delta^{(n_4)}$, $\sigma_{2, m} = -1$ and such that $P' \in Q_{1+2\delta^{(m)}}(z_{1, m})$. As n_2, n_4 are contained in the same

potentially δ_2 -bad component we conclude from Lemma 14 that $t_{1,m} < t_{1,n_4}$. By the non-existence of n_3 we also have $t_{1,m} > t_{1,n_2}$. But then either $\{z_{1,m}, z_{1,n_1}\}$ forms a spatially δ_2 -unstable pair (yielding a contradiction to Lemma 13) or we obtain a contradiction to the assumption $t_{1,n_4} = \text{height}_{(\varphi_1)_{P_0} \cap (\varphi_1)_{\delta_2}^{y_{1,n_1}}}(P_0)$.

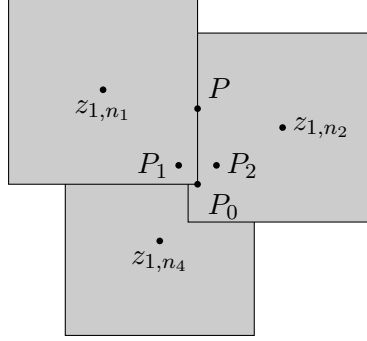


Figure 5: configuration in case (1a)

case 1b. Let us assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (0, \delta^{(n_1)} + \delta^{(n_2)}) \cup (1 - \delta^{(n_1)} - \delta^{(n_2)}, 1)$. However now assume additionally the existence of $1 \leq n_3 \leq N$ with $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_3)}$ -unstable pair. Among all these values choose n_3 with the property that $\pi_2(z_{1,n_3})$ is maximal. Observe that $\{z_{1,n_1}, z_{1,n_3}\}$ either forms a spatially δ_2 -unstable pair or $\{y_{1,n_1}, y_{1,n_2}, y_{1,n_3}\}$ forms a δ_2 -unstable triple. Therefore Lemma 13 implies $\sigma_{2,n_3} = 1$. Define $\xi_1 = (1 - 2 \cdot 1_{\pi_1(P) > \pi_1(z_{1,n_2})})$. Let us define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 - \delta^{(n_3)})e_2$ and $P_2 = (\pi_1(P) + \xi_1 \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 - \delta^{(n_3)})e_2$. See Figure 6 for an illustration.

- (i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ is clear.
- (ii) To show that P_i is $\psi^\Delta(n_i)$ -black assume for the sake of contradiction the existence of $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 14 we have $t_{1,m} < t_{1,n_2}$. By Lemma 13 and the choice of n_3 , we see that this is only possible if we have $\pi_2(z_{1,m}) \in (\pi_2(z_{1,n_3}) - \delta^{(m)} - \delta^{(n_3)}, \pi_2(z_{1,n_3}))$. But then again n_1, n_2, n_3 and m are contained in the same potentially δ_2 -bad connected component C therefore yielding a contradiction to Lemma 13.
- (iii) $|\pi_1(P) - \pi_1(P_i)| \leq \delta^{(n_1)}$ is clear.
- (iv) Suppose we could find $P' \in [P_1, P_2]$ and $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < t_{1,n_3} + \delta^{(n_3)}$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. As n_2, n_3 are contained in the same potentially δ_2 -bad component we conclude from Lemma 14 that $t_{1,m} < t_{1,n_3}$. Again by Lemma 13 and the choice of n_3 , we see that this is only possible if we have $\pi_2(z_{1,m}) \in (\pi_2(z_{1,n_3}) - (\delta^{(n_3)} + \delta^{(m)}), \pi_2(z_{1,n_3}))$. But again this would yield a contradiction to Lemma 13, since then n_1, n_2, n_3, m would be contained in the same potentially δ_2 -bad component.

case 2a. Henceforth we may assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$ and – due to case (1a) – that $\pi_2(z_{1,n_1}) - 1/2 < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_1)} - \delta^{(n_2)}$. Furthermore assume that there does *not* exist $1 \leq n_3 \leq N$ with $\sigma_{2,n_3} = -1$, $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_3}, z_{1,n_1}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_3)}$ -unstable pair (observe that in contrast to cases (1a) and (1b), we require also $\sigma_{2,n_3} = -1$). Let

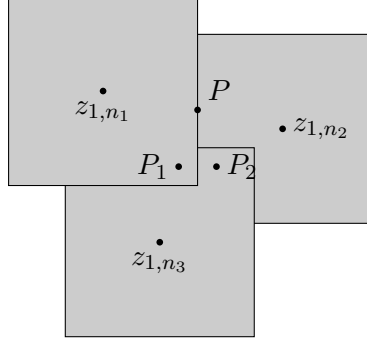


Figure 6: configuration in case (1b)

us define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)})e_2$ and $P_2 = (\pi_1(P) + \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)})e_2$. See Figure 7 for an illustration.

- (i) $P_1 \in Q_{1-2\delta^{(n_1)}}(z_{1,n_1})$ is clear and $P_2 \in Q_{1-2\delta^{(n_2)}}(z_{1,n_2})$ follows from our assumption $\pi_2(z_{1,n_1}) - 1/2 + \delta^{(n_1)} < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_2)}$ (observe that we additionally have $P_1 \in Q_{1-2\delta^{(n_2)}}(z_{1,n_2})$)
- (ii) To show that P_i is $\psi^\Delta(n_i)$ -black assume for the sake of contradiction the existence of $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 14 we have $t_{1,m} < t_{1,n_2}$. By Lemma 13 and the non-existence of n_3 , we see that this is impossible.
- (iii) $|\pi_1(P) - \pi_1(P_i)| \leq \delta^{(n_1)}$ is clear.
- (iv) Suppose we could find $P' \in [P_1, P_2]$ and $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < t_{1,n_2} + \delta^{(n_2)}$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. We conclude from Lemma 14 that $t_{1,m} < t_{1,n_2}$. Again by Lemma 13 and the non-existence of n_3 , we see that this is impossible.

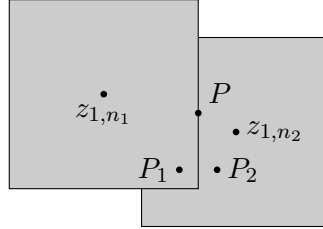


Figure 7: configuration in case (2a)

case 2b. Henceforth we may assume $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$ and – due to case (1a) – that $\pi_2(z_{1,n_1}) - 1/2 < \pi_2(z_{1,n_2}) + 1/2 - \delta^{(n_1)} - \delta^{(n_2)}$. Furthermore we may assume the existence of $1 \leq n_3 \leq N$ with $\sigma_{2,n_3} = -1$, $t_{1,n_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_1}, z_{1,n_3}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_3)}$ -unstable pair. Among all those values choose n_3 such that $\pi_2(z_{1,n_3}) + \delta^{(n_3)}$ is maximal. Furthermore in case (2b) we also assume that there does *not* exist $1 \leq n_4 \leq N$ with $\sigma_{2,n_4} = -1$, $t_{1,n_4} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n_4})| \leq 1/2$, $\pi_2(z_{1,n_4}) - 1/2 \in (\pi_2(z_{1,n_3}) + 1/2, \pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)} + 2\delta^{(n_4)})$ and such that $\{z_{1,n_4}, z_{1,n_1}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_4)}$ -unstable pair. Let us define $P_1 = (\pi_1(P) - \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)})e_2$ and $P_2 = (\pi_1(P) + \delta^{(n_1)})e_1 + (\pi_2(z_{1,n_3}) + 1/2 + 2\delta^{(n_3)})e_2$. See Figure 8 for an illustration.

- (i) $P_i \in Q_{1-2\delta^{(n_i)}}(z_{1,n_i})$ is clear, as otherwise $\{z_{1,n_2}, z_{1,n_3}\}$ would form a spatially $2\delta^{(n_2)} + 2\delta^{(n_3)}$ unstable pair, contradicting Lemma 13.
- (ii) To show that P_i is $\psi^\Delta(n_i)$ -black assume for the sake of contradiction the existence of $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < t_{1,n_i} + \delta^{(n_i)}$, $\sigma_{2,m} = -1$ and $P_i \in Q_{1+2\delta^{(m)}}(z_{1,m})$. Observe that by Lemma 14 we have $t_{1,m} < t_{1,n_2}$. By Lemma 13, the choice of n_3 and the non-existence of n_4 , we see that this is impossible.
- (iii) $|\pi_1(P) - \pi_1(P_i)| \leq \delta^{(n_i)}$ is clear.
- (iv) Suppose we could find $P' \in [P_1, P_2]$ and $1 \leq m \leq N$ with $t_{1,m} - \delta^{(m)} < \max(t_{1,n_1} + \delta^{(n_1)}, t_{1,n_2} + \delta^{(n_2)})$, $\sigma_{2,m} = -1$ and $P' \in Q_{1+2\delta^{(m)}}(z_{1,m})$. We conclude from Lemma 14 that $t_{1,m} < t_{1,n_2}$. Again by Lemma 13, the choice of n_3 and the non-existence of n_4 we see that this is impossible.

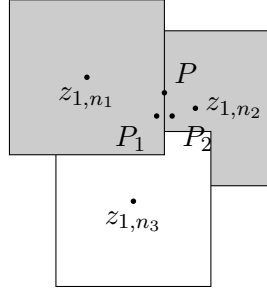


Figure 8: configuration in case (2b)

case 2c. We can now tackle the remaining case. By the previous cases we may work under the following assumptions

- $\pi_1(z_{1,n_2}) - \pi_1(z_{1,n_1}) \in (\delta^{(n_1)} + \delta^{(n_2)}, 1 - \delta^{(n_1)} - \delta^{(n_2)})$
- there exists $1 \leq n_3 \leq N$ with $t_{1,n_3} < t_{1,n_2}$, $\sigma_{2,n_3} = -1$, $|\pi_1(P) - \pi_1(z_{1,n_3})| \leq 1/2$, $\pi_2(z_{1,n_3}) + 1/2 \in (\pi_2(z_{1,n_1}) - 1/2, \pi_2(P))$ and such that $\{z_{1,n_3}, z_{1,n_1}\}$ does *not* form a $\delta^{(n_1)} + \delta^{(n_3)}$ -unstable pair.
- there exists $1 \leq n_4 \leq N$ with $t_{1,n_4} < t_{1,n_2}$, $\sigma_{2,n_4} = -1$, $|\pi_1(P) - \pi_1(z_{1,n_4})| \leq 1/2$, $\pi_2(z_{1,n_4}) - 1/2 \in (\pi_2(P), \pi_2(z_{1,n_2}) + 1/2)$.
- Furthermore if we choose n_3 so that $\pi_2(z_{1,n_3}) + \delta^{(n_3)}$ is maximal and n_4 so that $\pi_2(z_{1,n_4}) - \delta^{(n_4)}$ is minimal then we have $\pi_2(z_{1,n_4}) - \pi_2(z_{1,n_3}) \leq 1 + 2\delta^{(n_3)} + 2\delta^{(n_4)}$

See Figure 9 for an illustration. Observe that $\{z_{1,n_3}, z_{1,n_4}\}$ forms a spatially $2\delta^{(n_3)} + 2\delta^{(n_4)}$ -unstable pair. Furthermore we also make the following definitions

- choose $1 \leq n'_3 \leq N$ with $t_{1,n'_3} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n'_3})| \leq 1/2$, $\pi_2(z_{1,n'_3}) + 1/2 \in [\pi_2(z_{1,n_3}) + 1/2, \pi_2(P))$ and $\pi_2(z_{1,n'_3})$ is maximal
- choose $1 \leq n'_4 \leq N$ with $t_{1,n'_4} < t_{1,n_2}$, $|\pi_1(P) - \pi_1(z_{1,n'_4})| \leq 1/2$, $\pi_2(z_{1,n'_4}) - 1/2 \in (\pi_2(P), \pi_2(z_{1,n_4}) - 1/2]$ and $\pi_2(z_{1,n'_4})$ is minimal

As n_3 and n_4 have the properties required in the definition of n'_3 respectively n'_4 , this definition is indeed sensible (i.e. we are not choosing n'_3 or n'_4 from an empty set of possible values). Finally in this situation $\{y_{1,n'_3}, y_{1,n'_4}, y_{1,n_1}\}$ forms a δ_2 -unstable triple contradicting Lemma 13.

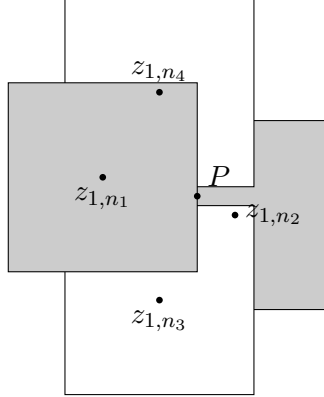


Figure 9: configuration in case (2c)

□

After these preliminaries the existence of ψ^Δ -black crossings of R_2 is rather immediate. Indeed suppose that Γ is a ψ_{φ_1} -black horizontal crossing of R_1 . Subdivide Γ into closed sub-paths $\Gamma_1, \dots, \Gamma_r$ such that if we denote by Γ_i^o the subset Γ_i obtained by deleting its two endpoints, then there exist $z_{n_1}, \dots, z_{n_r} \in Z$ with the property that $\text{height}_{\varphi_1}(\Gamma_i^o) = \{t_{n_i}\}$ for all $i \in \{1, \dots, r\}$ (see also Figure 10). Let us write $\{P_i\} = \Gamma_i \cap \Gamma_{i+1}$. Applying Lemma 16 to P_i and choosing $j_0 \in \{0, 1\}$ with $\text{height}_{\varphi_1}(P_i) = t_{n_i+j_0}$ we obtain for $j \in \{0, 1\}$ points $P_{i,j} \in \text{height}_{\varphi_1}^{-1}(t_{n_i+j} + \delta^{(n_i+j)})$ with the following properties

- let $k \in \{1, 2\}$ be such that $|\pi_k(P_i) - \pi_k(z_{1,n_i+j_0})| = 1/2$. Then for $j \in \{1, 2\}$ we have $|\pi_k(P_{i,j}) - \pi_k(P_i)| \leq \delta^{(n_i+j_0)}$
- the linear segment $P_{i,1}P_{i,2}$ is ψ^Δ -black

In particular we may apply Lemma 15 (with $n = i$, $P_1 = P_{i,2}$, $P_2 = P_{i+1,1}$, $P'_1 = P_i$, $P'_2 = P_{i+1}$) to obtain ψ^Δ -black paths from $P_{i,2}$ to $P_{i+1,1}$. Using the ψ^Δ -blackness of the joining segments $P_{i,1}P_{i,2}$ we can create a ψ^Δ -black path starting from $P_{i,1}$ and ending at $P_{i+1,2}$.

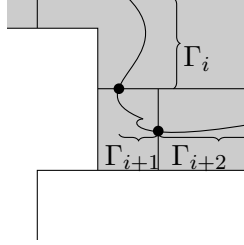


Figure 10: construction of the paths Γ_i

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